On the Embeddability of Graphs in Euclidean Spaces

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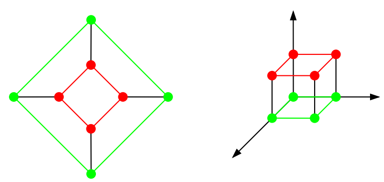

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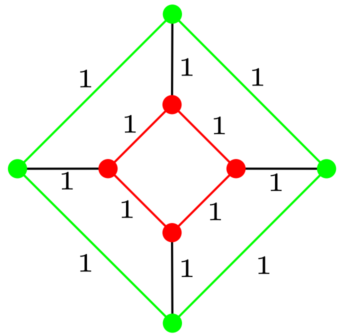
Outline

- Graph Embedding
- Facts for Embeddable Graphs
- Realizable Graphs
- Distance Matrices
- Embeddability of Almost Complete Graphs
- Conclusion – Open Problems

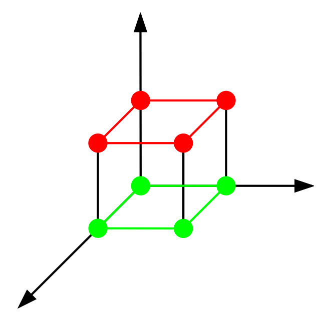
Graph Embedding

- Given is $G(V, E, w)$, an undirected edge weighted graph with $w_{ij} = w(e_{ij}) = w(v_i, v_j) > 0$.

- Points $p_1, p_2, \cdots, p_n \in \mathbb{R}^d$ form an embedding of G if and only if $|p_i - p_j| = w_{ij} \, \forall \, e_{ij} \in E$.



$G(V, E, w)$

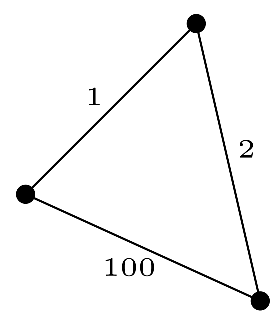


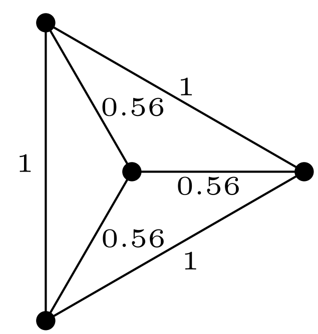
$G(V, E, w)$ embedded in $\mathbb{R}^3$

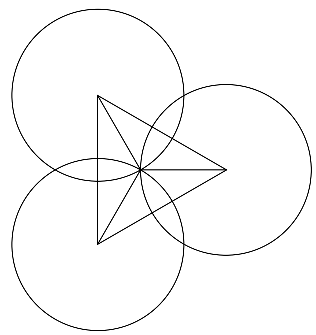
Non embeddable graphs in any space

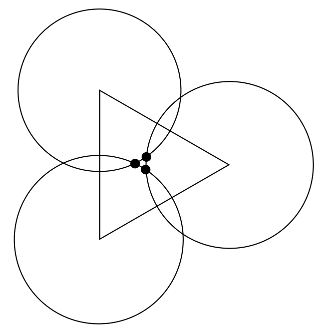
- All these graphs that violate the triangle inequality.

- Also there are graphs that don't violate the triangle inequality but cannot be embedded in any space.









- $a = \frac{\sqrt{3}}{3} \approx 0.577$ but the given edge weight is 0.56.

- In general take the d-dimensional 1-simplex, and a point attached to the barycenter. Assign to every edge that connects the barycenter to the simplex, a weight less than the real length of the corresponding line segment.

Facts for embeddable graphs

- If G is embeddable in $\mathbb{R}^d$ then $d \leq n-1$.

- There are cases where $d = n-1$ (G is embedded in $\mathbb{R}^{n-1}$ as a 1-simplex).

- If G is embeddable in $\mathbb{R}^d$ then it is embeddable in $\mathbb{R}^{d^*}$ where $d^* = \lfloor \frac{\sqrt{8|E|+1}-1}{2} \rfloor$ (Barvinok, 1995).

Realizable Graphs

- A graph $G(V, E)$ is realizable in $\mathbb{R}^d$ iff G is embeddable in $\mathbb{R}^d$ for any choice of edge weights.

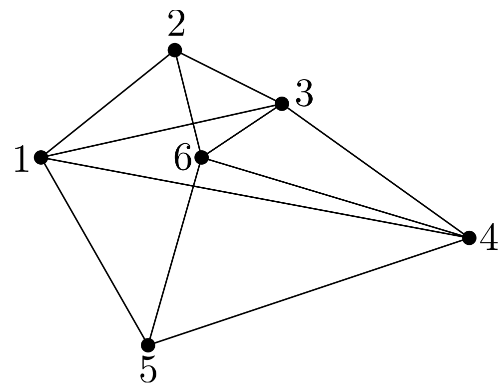
- For example $G = K_4$ is not 2-realizable because G cannot be embedded in $\mathbb{R}^2$, for any choice of edge weights (consider the case of $w(e) = 1$ for every edge).

- Recently (2004) Bob Connely, completely characterized realizable graphs in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Partial Euclidean Distance Matrix

- Let $G(V, E, w)$ be an undirected edge weighted graph.

- The $n \times n$ matrix $D_P = (d_{ij}) = w_{ij}^2$ is called the Partial Euclidean Distance Matrix associated with G.
  - D_P is symmetric,
  - has zero diagonal,
  - $d_{ij} = w_{ij}^2 \, \forall \, e_{ij} \in E$, otherwise d_{ij} is undefined.

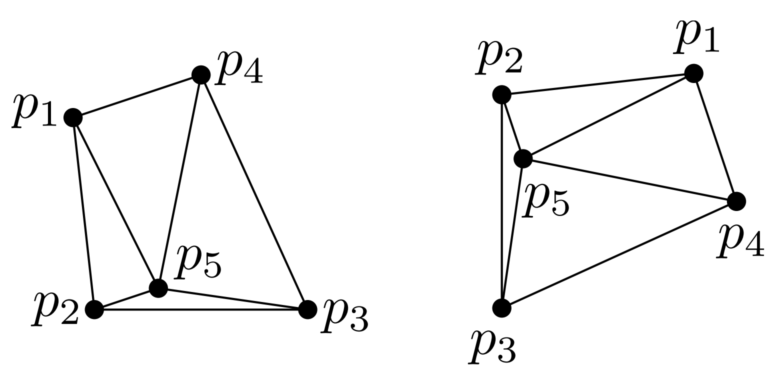


$$D_P = \begin{bmatrix} 0 & a & b & c & d & ? \\ a & 0 & e & ? & ? & f \\ b & e & 0 & g & ? & h \\ c & ? & g & 0 & i & j \\ d & ? & ? & i & 0 & f \\ ? & f & h & j & f & 0 \end{bmatrix}$$

Congruent Embeddings

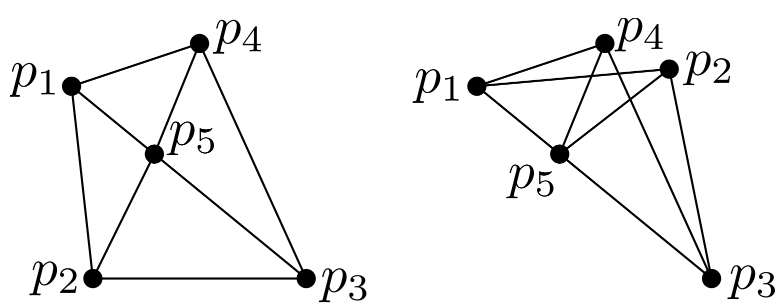
- Two embeddings of the same graph are congruent if they have identical Distance Matrices.

- The Distance Matrix is invariant under distance preserving transformations such as translations, rotations and space inversions.



Non Congruent Embeddings

- Two embeddings of the same graph with different Distance Matrices.



Euclidean Distance Matrix Completion Problem

- Assign values to the undefined elements of D_P in a way that there exist points $p_1, p_2, \cdots, p_n \in \mathbb{R}^d$ such that $d_{ij} = \|p_i - p_j\|^2$.

- This problem is equivalent to the Graph Embedding prolbem:

  Graph $G(V, E, w)$ is embeddable if and only if the graph's partial Distance Matrix can be completed to a Euclidean Distance Matrix.

Conditions for Euclidean Distance Matrices

- Matrix D is a Euclidean Distance Matrix if and only if matrix X:

$$x_{ij} = \frac{1}{2}(d_{in} + d_{jn} - d_{ij}) \text{ for all } i, j = 1, \cdots, n-1$$

  is positive semidefinite.

- The smallest embedding dimension of D is rank(X).

- The embeddability of a an u.e.w complete graph $G = K_n$ can be effectively computed.

Arbitrary Graphs

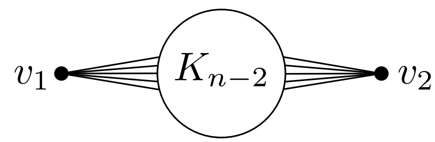
- The d-embeddability problem is NP-hard. Remains NP-hard even when the edge weights are all restricted to 1 or 2.

- Uniqueness of embeddings is also NP-hard.

- The complexity of the Embeddability problem is not settled.

Embeddability of almost complete graphs

The embeddability of an almost complete graph that misses one edge is $\max(d_1, d_2)$.

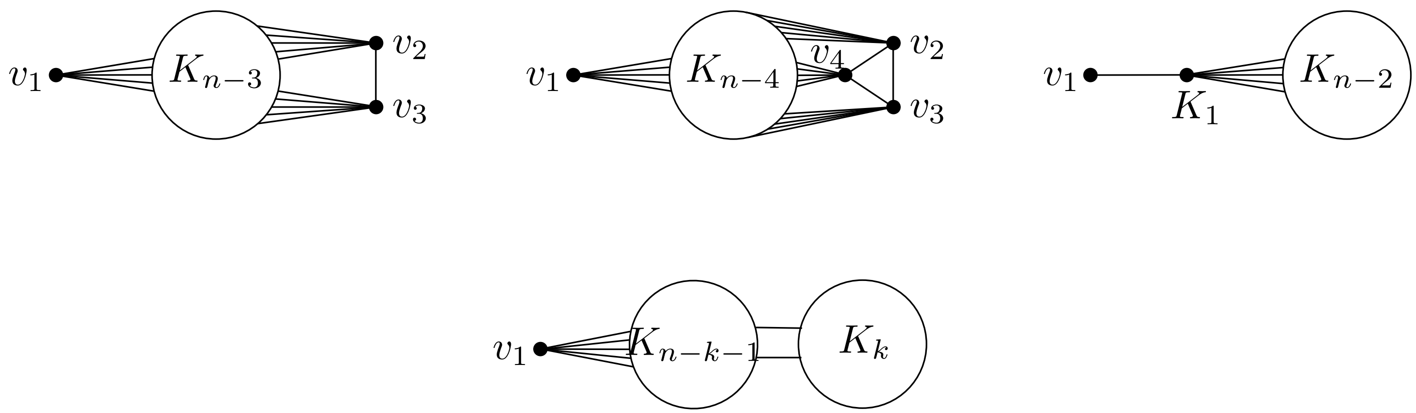


- Kernel $\cup\, v_1$ embedded in $\mathbb{R}^{d_1}$. Kernel $\cup\, v_2$ embedded in $\mathbb{R}^{d_2}$

- Assume $d_1 \leq d_2$, and extend the d_1-embedding to $\mathbb{R}^{d_2}$.

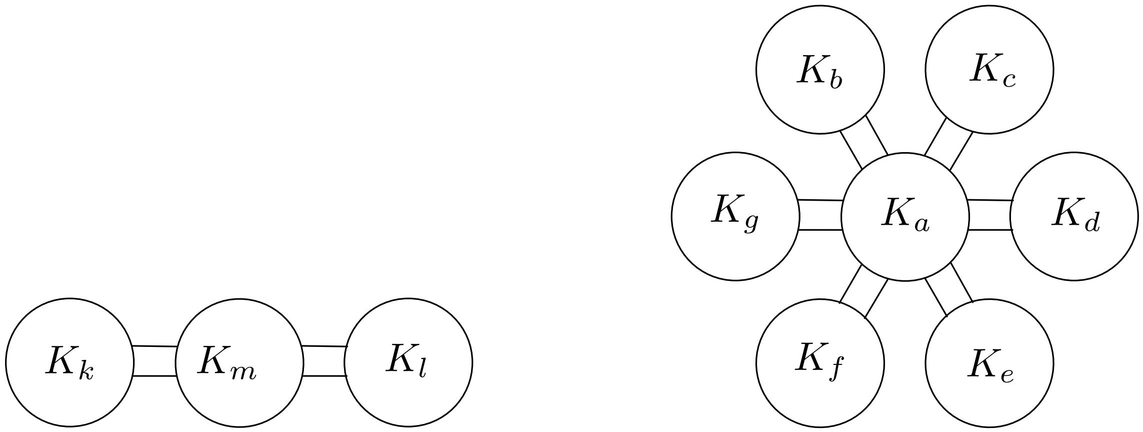
- Draw $n-2$ hyperspheres. Take as v_2, a common point of the hyperspheres.

Embeddability of almost complete graphs (cont.)

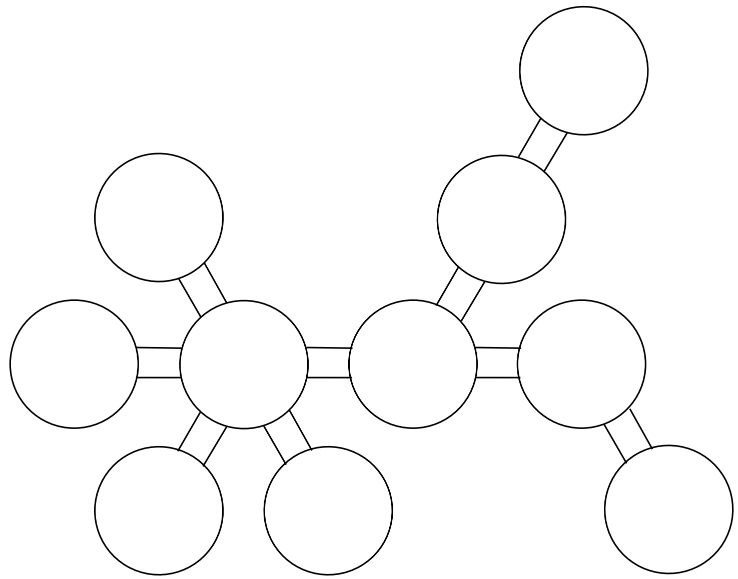


The embeddability of an almost complete graph that misses k edges that are all incident to one vertex is max(d_1, d_2).

Star of cliques



Tree of cliques



Conclusion – Open Problems

- Characterize the embeddability problem in other graph topologies.